Differential Geometry IV: General Relativity

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10 Sep. 2025

1.1 Let V be an n-dimensional vector space and $m : V \times V \to \mathbb{R}$ a Lorentzian inner product on V. Recall that, for any timelike vector $v \in V$, we have defined

$$|v| \doteq \sqrt{-m(v,v)}$$

(a) Let $v \in V$ be a *timelike* vector in V. Show that the hyperplane

$$v^\perp \doteq \{w \in V : m(v,w) = 0\}$$

is a spacelike subspace of V.

(b) Show that that, for any two timelike vectors $v, w \in V$, the inverse Cauchy–Schwarz inequality

$$|m(v,w)| \geqslant |v||w|$$

and (in the case when v, w belong to the same component of the timelike cone) the inverse triangle inequality

$$|v + w| \geqslant |v| + |w|$$

hold, with equality only in the case when v and w are colinear.

1.2 Let V be an $(n+1)$-dimensional vector space equipped with a Lorentzian inner product m.

(a) Prove that any two *null* vectors v, w of V that are orthogonal are also *collinear*.

(b) Prove that if v and w are *causal* vectors that are orthogonal, then they have to be *null* and *collinear*.

(c) Prove that if v is a *null* vector, then its orthogonal complement

$$v^\perp = \{w \in V : m(v,w) = 0\}$$

is a null hyperplane containing v.

1.3 Let $\mathcal{M}$ be a differentiable manifold of dimension n and $p \in \mathcal{M}$. Recall that the tangent space $T_p\mathcal{M}$ at p is defined as the set of all functionals $X : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying the product rule

$$X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g).$$

Prove that the set $T_p\mathcal{M}$ is a vector space of dimension n. (*Hint: Use the fact that, in any given local coordinate chart $\phi : \mathcal{U} \to \phi(\mathcal{U}) \subset \mathbb{R}^n$ on a neighborhood $\mathcal{U}$ around p with $\phi(p) = 0$, any smooth function $f : \phi(\mathcal{U}) \to \mathbb{R}$ can be expanded as $f(x) = f(0) + A_a x^a + B_{ab}(x)x^a x^b$ for constants $\{A_a\}_{a=1}^n$ and smooth functions $\{B_{ab}(x)\}_{a,b=1}^n$.*)

1.4 Let $\mathcal{M}^n$ be a differentiable manifold and V be a smooth vector field on M. Assume that $V(p) \neq 0$ for some $p \in \mathcal{M}$. Show that there exists an open neighborhood $\mathcal{U}$ of p and a coordinate chart $(x^1, \ldots, x^n)$ on $\mathcal{U}$ such that $V = \frac{\partial}{\partial x^1}$ in $\mathcal{U}$.

1.5 Let X, Y, Z be smooth vector fields on a differentiable manifold $\mathcal{M}$. We define the commutator (or *Lie bracket*) $[X, Y]$ of X and Y to be the vector field satisfying for any function $f \in C^\infty(\mathcal{M})$

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

(a) Show that $[X, Y]$ satisfies the following identities:

1. $[X, Y] = -[Y, X]$ (*anticommutativity*).
2. $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for any constans a, b (*linearity*).
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*).

(b) Let X^a and Y^b be the components of X and Y, respectively, in a local coordinate chart $(x^1, \ldots, x^n)$ on $\mathcal{M}$ (i.e. $X = X^a \frac{\partial}{\partial x^a}$ and $Y = Y^a \frac{\partial}{\partial x^a}$). Compute the components of $[X, Y]$ in the same coordinate chart.